TRY THIS AT HOME

Quadratic Equations - Four ways to solve

In our podcast we mentioned several ways of solving quadratic equations, including a “new” way based on the work of Po Shen Loh (who credits the work of Viète, the Babylonians and the Greeks). He posts updates to his method on his website if you want to read about it in his words.

First, what is a quadratic equation? A quadratic equation is the equation for a parabola which is a U shaped curve. The curve can open either up or down. The name Quadratic comes from “quad” meaning square, because the variable gets squared (like x^2). The standard form of the equation looks like this:

$$ax^2 + bx + c = 0$$

Where a, b and c are known values, and $a \neq 0$.

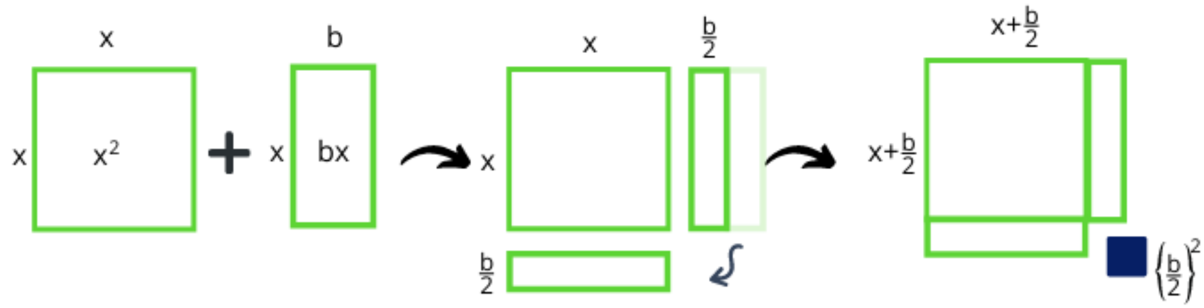
■ Trial and Error Factoring

The first way we will solve is to factor the equation into two expressions that when multiplied together will give you the quadratic. Typically, this is taught as. trial and error where you try to find two numbers that satisfy the following: sum = b; product = c. Sometimes this is easier than others. Try to solve the following quadratic equations using trial and error factoring.

1. $u^2 - 5u - 14 = 0$
2. $x^2 + 15x + 50 = 0$
3. $y^2 - 11y + 28 = 0$Q

■ Completing the Square

The next method we will use is called completing the square. To understand completing the square, let’s start with a simple expression like $x^2 + bx$. All of those x's make the expression hard to solve so we can use geometry to think about it this way.



This method is called completing the square because you can see that $x2 + bx$ is almost a square and we just need to add $(b/2)^2$ to make it a complete square. When we rearrange using algebra we get:

$$x^2 + bx + (b/2)^2 = (x + b/2)^2$$

Let's see how this works with an example:

$$x^2 + 6x + 7 = 0$$

(the $b = 6$, so that means $b/2 = 6/2 = 3$)

We need to add $(b/2)^2$ to each side of the equation.

$$x^2 + 6x + (b/2)^2 + 7 = 0 + (b/2)^2$$

A little substitution and algebra get us:

$$\begin{aligned} x^2 + 6x + 3^2 + 7 &= 0 + 3^2 \\ (x^2 + 6x + 3^2) + 7 &= 9 \\ (x^2 + 6x + 3^2) &= 2 \\ (x + 3)^2 &= 2 \\ x + 3 &= \pm\sqrt{2} \\ x &= -3 \pm \sqrt{2} \end{aligned}$$

Try the complete the square method to solve these quadratic equations:

4. $x^2 + 3x - 10 = 0$
5. $z^2 - 12z + 4 = 0$
6. $t^2 - 7t + 2 = 0$

■ The Quadratic Formula

The quadratic formula gives you wa way to solve equations where factoring may not be easy to use. It does require you to remember a long formula and the formula contains square roots and fractions which can be intimidating. Let's recall what a quadratic equation is. We will use the values for a, b and c in the quadratic formula.

$$ax^2 + bx + c = 0$$

Where a, b and c are known values, and $a \neq 0$.

The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So if we substitute the values for a, b and c into the quadratic formula we can solve for x. Try it!

7. $x^2 - 4x + 2 = 0$
8. $3x^2 + x - 10 = 0$
9. $2y^2 - 8y + 5 = 0$

■ Factoring Without Trial and Error - The New Method

On the podcast we talked about how the method from Po Shen Ho allows the ease of factoring for a solution without having have integers that add and multiply to a specific value. This method also makes it possible to factor and still get irrational and imaginary roots. So we need to make a couple of modification so the trial and error method. When the value of the coefficient $a \neq 1$, it is trickier to factor. For this method, we start by dividing all of the values by a to remove the coefficient which changes the other coefficients such that $\frac{b}{a} = B, \frac{c}{a} = C$.

Similar to the original factoring method we need to find values such that the sum $= -B$ and the product $= C$. However, instead of guessing and trying we use the concept that $\frac{B}{2}$ is actually the average of the two values we are looking for. And because it is only two values, we know that we need to add or subtract some common value from the average in order to get our answers. So using this information, we can substitute into the equation for the product:

$$(\frac{-B}{2} + u)(\frac{-B}{2} - u) = C$$

With some algebra, the left hand side of the equation can be simplified to:

$$\frac{-B^2}{4} - u^2 = C$$

And then rearranged to:

$$u^2 = \frac{-B^2}{4} - C$$

We can then solve for u because we know the values for B and C. Then we substitute u back into the expressions for the factors and solve.

$$(\frac{-B}{2}+u),(\frac{-B}{2}-u)$$

Try this new factoring method for yourself!

10. $x^2 - 2x - 24 = 0$
11. $m^2 + 4m - 20 = 0$
12. $5x^2 - 7x - 1 = 0$

Solutions.

1. $(u-7)(u+2) = 0$
   $u = -2, u = 7$

2. $(x+5)(x+10) = 0$
   $x = -5, x = -10$

3. $(x-4)(x-7) = 0$
   $x = 4, x = 7$

4. $x^2 + 3x + \left(\frac{3}{2}\right)^2 = 10 + \left(\frac{3}{2}\right)^2$
   $\left(x+\frac{3}{2}\right)^2 = \frac{49}{4}$
   $x + \frac{3}{2} = \pm\frac{7}{2}$
   $t - \frac{7}{2} = \pm\frac{\sqrt{41}}{2}$
   $t = \frac{7}{2} \pm \frac{\sqrt{41}}{2}$
   $t = \frac{7}{2} + \frac{\sqrt{41}}{2}, t = \frac{7}{2} - \frac{\sqrt{41}}{2}$
   $x = -5, x = 2$

5. $z^2 - 12z + \left(\frac{-12}{2}\right)^2 = -4 + \left(\frac{-12}{2}\right)^2$
   $(z-6)^2 = 32$
   $z - 6 = \pm 4\sqrt{2}$
   $z = 6 + 4\sqrt{2}, z = 6 - 4\sqrt{2}$

6. $t^2 - 7t + \left(\frac{-7}{2}\right)^2 = -2 + \left(\frac{-7}{2}\right)^2$
   $\left(t-\frac{7}{2}\right)^2 = -2 + \frac{49}{4}$
   $\left(t-\frac{7}{2}\right)^2 = \frac{41}{4}$

7. $x = \dfrac{4 \pm \sqrt{(-4)^2 - (4)(1)(2)}}{(2)(1)}$
   $x = \dfrac{4 \pm \sqrt{8}}{2}$
   $x = \dfrac{4 \pm 2\sqrt{2}}{2}$

$x = 2 \pm \sqrt{2}$
$x = 2 + \sqrt{2}, x = 2 - \sqrt{2}$

8. $x = \dfrac{-1 \pm \sqrt{(1)^2 - (4)(3)(-10)}}{(2)(3)}$
$x = \dfrac{-1 \pm \sqrt{1 + 120}}{6}$
$x = \dfrac{-1 \pm 11}{6}$
$x = \dfrac{5}{3}, x = 2$

9. $y = \dfrac{8 \pm \sqrt{(-8)^2 - (4)(2)(5)}}{(2)(2)}$
$y = \dfrac{8 \pm \sqrt{64 - 40}}{4}$
$y = \dfrac{4 \pm \sqrt{6}}{2}$
$y = \dfrac{4 + \sqrt{6}}{2}, y = \dfrac{4 - \sqrt{6}}{2}$

10. $x^2 - 2x - 24 = 0$
$\left(\dfrac{2}{2} + u\right)\left(\dfrac{2}{2} - u\right) = -24$
$(1 + u)(1 - u) = -24$
$1 - u^2 = -24$
$u^2 = 25$
$u = 5$
$x = 6, x = -4$

11. $m^2 + 4m - 20 = 0$
$\left(\dfrac{-4}{2} + u\right)\left(\dfrac{-4}{2} - u\right) = -20$
$(-2 + u)(-2 - u) = -20$
$4 - u^2 = -20$
$u^2 = 24$
$u = 2\sqrt{6}$

$m = -2 + 2\sqrt{6}, m = -2 - 2\sqrt{6}$

12. $x^2 + 2x - \dfrac{1}{3} = 0$
$\left(\dfrac{-2}{2} + u\right)\left(\dfrac{-2}{2} - u\right) = -\dfrac{1}{3}$
$(-1 + u)(-1 - u) = -\dfrac{1}{3}$
$1 - u^2 = -\dfrac{1}{3}$
$u^2 = \dfrac{4}{3}$
$u = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$

$x = -1 + \dfrac{2\sqrt{3}}{3}, x = -1 - \dfrac{2\sqrt{3}}{3}$